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Differential geometry of the Z_3-graded quantum superplane

Salih Çelik

Yildiz Technical University, Department of Mathematics, 34210 Davutpasa-Esenler, Istanbul, Turkey

E-mail: sacelik@yildiz.edu.tr

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Abstract
In this work, differential geometry of the Z_3-graded quantum superplane is constructed. The corresponding quantum Lie superalgebra and its Hopf algebra structure are obtained.

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1. Introduction

Noncommutative geometry [1] has started to play an important role in different fields of mathematical physics over the past decade. The basic structure giving a direction to the noncommutative geometry is a differential calculus on an associative algebra. The noncommutative differential geometry of quantum groups was introduced by Woronowicz [2]. In this approach, the quantum group is taken as the basic noncommutative space and the differential calculus on the group is deduced from the properties of the group. The other approach, initiated by Wess and Zumino [3], followed Manin's emphasis [4] on the quantum spaces as the primary objects. Differential forms are defined in terms of noncommuting (quantum) coordinates, and the differential and algebraic properties of quantum groups acting on these spaces are obtained from the properties of the spaces. The natural extension of their scheme to superspace [5] was introduced in [6, 7].

Recently, there have been many attempts to generalize Z_2-graded constructions to the Z_3-graded case [8–12]. Chung [12] studied the Z_3-graded quantum space that generalizes the Z_2-graded space called a superspace, using the methods of Wess and Zumino [3]. In this work, we have investigated the noncommutative geometry of the Z_3-graded quantum superplane. These calculi are discussed from the covariance point of view, using the Hopf algebra structure of the quantum superplane [13]. In order to obtain the corresponding quantum Z_3-grading Lie superalgebra, we constructed a left-covariant differential calculus on the Z_3-graded quantum superplane (of course, this may also be done using a right-covariant differential calculus on it). Hopf algebra structure of the obtained superalgebra is given, using the method in [14].

Let us briefly investigate a general Z_3-graded algebraic structure. Let z be a Z_3-graded variable. Then we say that the variable z satisfies the relation

$$z^3 = 0.$$

If $f(z)$ is an arbitrary function of the variable z, then the function $f(z)$ becomes a polynomial of degree two in z, that is,

$$f(z) = a_0 + a_1 z + a_2 z^2$$

where a_0, a_2, a_1 denote three fixed numbers whose grades are $\text{grad}(a_0) = 0$, $\text{grad}(a_2) = 1$ and $\text{grad}(a_1) = 2$, respectively.

The cyclic group Z_3 can be represented in the complex plane by means of the cubic roots of 1: let $j = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$ ($\mathrm{i}^2 = -1$). Then one has

$$j^3 = 1 \qquad \text{and} \qquad j^2 + j + 1 = 0 \qquad \text{or} \qquad (j+1)^2 = j.$$

One can define the Z_3-graded commutator $[A, B]$ as

$$[A, B]_{Z_3} = AB - j^{ab} BA$$

where $\text{grad}(A) = a$ and $\text{grad}(B) = b$. If A and B are j-commutative, then we have

$$AB = j^{ab} BA.$$

2. The algebra of functions on the Z_3-graded quantum superplane

It is well known that the Z_2-graded quantum plane or the quantum superplane is defined as an associative algebra whose even coordinate x and odd (Grassmann) coordinate θ satisfy

$$x\theta = q\theta x \qquad \theta^2 = 0$$

where q is a nonzero complex deformation parameter.

One of the possible ways to generalize the quantum superplane is to increase the power of nilpotency of its odd generator. So, a possible generalization can be defined as an associative unital algebra generated by x and θ satisfying

$$x\theta = q\theta x \qquad \theta^3 = 0. \tag{1}$$

Here, the coordinate x with respect to the Z_3-grading is of grade 0 and the coordinate θ with respect to the Z_3-grading is of grade 1.

The quantum superplane underlies a noncommutative differential calculus on a smooth manifold with exterior differential d satisfying $\mathsf{d}^2 = 0$. So the above mentioned generalization of the superplane raises the natural question of possible generalization of differential calculus to one with exterior differential d satisfying $\mathsf{d}^3 = 0$. From an algebraic point of view, a sufficient algebraic structure underlying a differential calculus is the notion of the Z_3-graded differential algebra. Therefore, we can generalize the differential calculus with the help of an appropriate generalization of Z_3-graded differential algebra.

Elementary properties of the Z_2-graded quantum superplane are described in [13]. We state briefly the properties we are going to need in this work.

Let $\mathcal{A}$ be a free unital associative algebra generated by two elements x, θ obeying relations (1). We know that the algebra $\mathcal{A}$ is a graded Hopf algebra with the following costructures [13]: the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is defined by

$$\Delta(x) = x \otimes x \qquad \Delta(\theta) = \theta \otimes x + x \otimes \theta \qquad \Delta(1) = 1 \otimes 1. \tag{2}$$

The counit $\epsilon : \mathcal{A} \to \mathcal{C}$ is given by

$$\epsilon(x) = 1 \qquad \epsilon(\theta) = 0. \tag{3}$$

We extend the algebra $\mathcal{A}$ by including the inverse of x which obeys

$$xx^{-1} = 1 = x^{-1}x.$$

If we extend the algebra $\mathcal{A}$ by adding the inverse of x then the algebra $\mathcal{A}$ admits a coinverse $S : \mathcal{A} \to \mathcal{A}$ defined by

$$S(x) = x^{-1} \qquad S(\theta) = -x^{-1}\theta x^{-1}. \tag{4}$$

Note that

$$\Delta(x^{-1}) = x^{-1} \otimes x^{-1}.$$

It is not difficult to verify the following properties of costructures:

$$\begin{aligned}
&(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \\
&\mu \circ (\epsilon \otimes \mathrm{id}) \circ \Delta = \mu' \circ (\mathrm{id} \otimes \epsilon) \circ \Delta \\
&m \circ (S \otimes \mathrm{id}) \circ \Delta = \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta
\end{aligned} \tag{5}$$

where id denotes the identity mapping,

$$\mu : \mathcal{C} \otimes \mathcal{A} \to \mathcal{A} \qquad \mu' : \mathcal{A} \otimes \mathcal{C} \to \mathcal{A}$$

are the canonical isomorphisms, defined by

$$\mu(c \otimes a) = ca = \mu'(a \otimes c) \qquad \forall a \in \mathcal{A} \quad \forall c \in \mathcal{C}$$

and m is the multiplication map:

$$m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \qquad m(a \otimes b) = ab. \tag{6}$$

The multiplication in $\mathcal{A} \otimes \mathcal{A}$ is defined with the rule

$$(A \otimes B)(C \otimes D) = j^{\mathrm{grad}(B)\mathrm{grad}(C)} AC \otimes BD. \tag{7}$$

This type of relation is called a braiding relation. The differential structure of braided quantum spaces is given in [15].

3. Differential calculi on the Z_3-graded quantum superplane

In this section, we shall build up the noncommutative differential calculus on the Z_3-graded quantum superplane. This involves functions on the superplane, differentials and differential forms. So we have to define a linear operator d which acts on the functions of the coordinates of the Z_3-graded quantum superplane. For the definition, it is sufficient to define the action of d on the coordinates and on their products.

We postulate that the linear operator d applied to x produces a 1-form whose Z_3-grade is 1 by definition. Similarly, application of d to θ produces a 1-form whose Z_3-grade is 2. We shall denote the obtained quantities by $\mathsf{d}x$ and $\mathsf{d}\theta$, respectively. When the linear operator d is applied to $\mathsf{d}x$ (or twice by iteration to x) it will produce a new entity which we shall call a 1-form of grade 2, denoted by d^2x and applied to $\mathsf{d}\theta$ produces a 1-form of grade 0, modulo 3, denoted by $\mathsf{d}^2\theta$. Finally, we require that $\mathsf{d}^3 = 0$.

3.1. *Differential algebra*

Let us begin the ordering the properties of the exterior differential. The exterior differential d is an operator which gives the mapping from the generators of the Z_3-graded quantum superplane to the differentials

$$\mathsf{d} : a \mapsto \mathsf{d}a \qquad a \in \{x, \theta\}.$$

We demand that the exterior differential d has to satisfy two properties:

$$\mathsf{d}^3 = 0 \tag{8}$$

and the Z_3-graded Leibniz rule

$$\mathsf{d}(fg) = (\mathsf{d}f)g + j^{\mathrm{grad}(f)} f(\mathsf{d}g). \tag{9}$$

It is well known that in classical differential calculus, functions commute with differentials. From an algebraic point of view, the space of 1-forms is a free finite bimodule over the algebra of smooth functions generated by the first-order differentials and the commutativity shows how its left and right structures are related to each other.

In order to establish a noncommutative differential calculus on the Z_3-graded quantum superplane, we assume that the commutation relations between the coordinates and their differentials are in the following form:

$$\begin{aligned}
x\,\mathsf{d}x &= X\,\mathsf{d}x\,x\\
x\,\mathsf{d}\theta &= A\,\mathsf{d}\theta\,x + B\,\mathsf{d}x\,\theta\\
\theta\,\mathsf{d}x &= C\,\mathsf{d}x\,\theta + D\,\mathsf{d}\theta\,x\\
\theta\,\mathsf{d}\theta &= Y\,\mathsf{d}\theta\,\theta.
\end{aligned} \tag{10}$$

The coefficients A, B, etc will be determined in terms of the complex deformation parameters q and j. To find them, we shall use the covariance of the noncommutative differential calculus.

Since we assume that $\mathsf{d}^3 = 0$ and $\mathsf{d}^2 \neq 0$, in order to construct a self-consistent theory of differential forms it is necessary to add to the first-order differentials of coordinates $\mathsf{d}x$, $\mathsf{d}\theta$ a set of *second-order differentials* d^2x, $\mathsf{d}^2\theta$. Appearance of higher order differentials is a peculiar property of a proposed generalization of differential forms. This has as a consequence certain problems.

Now, we assume that d is no longer the classical exterior differential, i.e. $\mathsf{d}^2 \neq 0$. For example, if we take a particular 1-form $\theta\,\mathsf{d}x$ and apply to it the exterior differential d, we obtain

$$\mathsf{d}(\theta\,\mathsf{d}x) = \mathsf{d}\theta\,\mathsf{d}x + j\theta\,\mathsf{d}^2x.$$

Therefore, differentiating (10) with regard to the Z_3-graded Leibniz rule (9) one gets

$$\begin{aligned}
x\,\mathsf{d}^2x &= X\,\mathsf{d}^2x\,x + (jX - 1)(\mathsf{d}x)^2\\
x\,\mathsf{d}^2\theta &= A\,\mathsf{d}^2\theta\,x + B\,\mathsf{d}^2x\,\theta + (j^2A + jBF - F)\,\mathsf{d}\theta\,\mathsf{d}x\\
\theta\,\mathsf{d}^2x &= j^{-1}C\,\mathsf{d}^2x\,\theta + j^{-1}D\,\mathsf{d}^2\theta\,x + (jD + CF - j^{-1})\,\mathsf{d}\theta\,\mathsf{d}x\\
\theta\,\mathsf{d}^2\theta &= Yj^{-1}\,\mathsf{d}^2\theta\,\theta + (jY - j^{-1})(\mathsf{d}\theta)^2.
\end{aligned} \tag{11a}$$

Here, we have assumed that

$$\mathsf{d}x\,\mathsf{d}\theta = F\,\mathsf{d}\theta\,\mathsf{d}x \qquad (\mathsf{d}x)^3 = 0 \tag{11b}$$

where F is a parameter that shall be described later.

Relations (11a) are *not homogeneous* in the sense that the commutation relations between the coordinates and second-order differentials include first-order differentials as well. Later, we shall see that the commutation relations between the coordinates and their second-order differentials can be made homogeneous. They will not include first-order differentials by removing them using the covariance of the noncommutative differential calculus.

We now return to relations (11a). Applying the exterior differential d to relations (11a), we obtain

$$\begin{aligned}
\mathrm{d}x\,\mathrm{d}^2x &= j^{-2}\,\mathrm{d}^2x\,\mathrm{d}x \\
\mathrm{d}x\,\mathrm{d}^2\theta &= -\frac{A}{Q}\,\mathrm{d}^2\theta\,\mathrm{d}x + \frac{1+B-j^2AF^{-1}}{Q}\,\mathrm{d}^2x\,\mathrm{d}\theta \\
\mathrm{d}\theta\,\mathrm{d}^2x &= -\frac{C}{Q'}\,\mathrm{d}^2x\,\mathrm{d}\theta + \frac{D-CF+j^{-1}}{Q'}\,\mathrm{d}^2\theta\,\mathrm{d}x \\
\mathrm{d}\theta\,\mathrm{d}^2\theta &= \mathrm{d}^2\theta\,\mathrm{d}\theta
\end{aligned} \tag{11c}$$

where

$$Q = AF^{-1} + j^2(1+B) \quad \text{and} \quad Q' = D + j^2(1+CF).$$

As a consequence, the second-order differentials have to satisfy the following relation:

$$\mathrm{d}^2x\,\mathrm{d}^2\theta = jF\,\mathrm{d}^2\theta\,\mathrm{d}^2x. \tag{11d}$$

3.2. Covariance

We see from the above relations (11a) that the commutation relations between the generators of $\mathcal{A}$ and their second-order differentials are *not homogeneous* in the sense that they include first-order differentials. In order to homogenize relations (11a), we shall consider the covariance of the noncommutative differential calculus.

We first note that consistency of a differential calculus with commutation relations (1) means that the differential algebra is a graded associative algebra generated by the elements of the set $\{x, \theta, \mathrm{d}x, \mathrm{d}\theta, \mathrm{d}^2x, \mathrm{d}^2\theta\}$.

Let $\Omega(\mathcal{A})$ be a free left module over the algebra $\mathcal{A}$ generated by the elements of the set $\{x, \theta, \mathrm{d}x, \mathrm{d}\theta, \mathrm{d}^2x, \mathrm{d}^2\theta\}$. The module $\Omega(\mathcal{A})$ becomes a unital associative algebra if one defines a multiplication law on $\Omega(\mathcal{A})$ by relations (1) and (11).

We consider a map $\phi_L : \Omega(\mathcal{A}) \to \mathcal{A} \otimes \Omega(\mathcal{A})$ such that

$$\phi_L \circ \mathrm{d} = (\tau \otimes \mathrm{d}) \circ \Delta \tag{12}$$

where $\tau : \Omega(\mathcal{A}) \to \Omega(\mathcal{A})$ is the linear map of degree zero which gives

$$\tau(a) = j^{\mathrm{grad}(a)}a \qquad \forall a \in \Omega(\mathcal{A}). \tag{13}$$

Thus we have

$$\phi_L(\mathrm{d}x) = x \otimes \mathrm{d}x \qquad \phi_L(\mathrm{d}\theta) = j\theta \otimes \mathrm{d}x + x \otimes \mathrm{d}\theta. \tag{14}$$

We now define a map Δ_L as follows:

$$\Delta_L(a_1\,\mathrm{d}b_1 + \mathrm{d}b_2a_2) = \Delta(a_1)\phi_L(\mathrm{d}b_1) + \phi_L(\mathrm{d}b_2)\Delta(a_2). \tag{15}$$

We now apply the linear map Δ_L to relations (15):

$$\begin{aligned}
\Delta_L(x\,\mathrm{d}x) &= \Delta(x)\phi_L(\mathrm{d}x) = X\phi_L(\mathrm{d}x)\Delta(x) = X\Delta_L(\mathrm{d}x\,x) \\
\Delta_L(x\,\mathrm{d}\theta) &= \Delta(x)\phi_L(\mathrm{d}\theta) \\
&= A\phi_L(\mathrm{d}\theta)\Delta(x) + B\phi_L(\mathrm{d}x)\Delta(\theta) + j(X - q^{-1}A - B)x\theta \otimes \mathrm{d}x\,x \\
&= A\Delta_L(\mathrm{d}\theta\,x) + B\Delta_L(\mathrm{d}x\,\theta) + j(X - q^{-1}A - B)x\theta \otimes \mathrm{d}x\,x \\
\Delta_L(\theta\,\mathrm{d}x) &= \Delta(\theta)\phi_L(\mathrm{d}x) \\
&= C\phi_L(\mathrm{d}x)\Delta(\theta) + D\phi_L(\mathrm{d}\theta)\Delta(x) + (X - qjC - jD)\theta x \otimes \mathrm{d}x\,x \\
&= C\Delta_L(\mathrm{d}x\,\theta) + D\Delta_L(\mathrm{d}\theta\,x) + (X - qjC - jD)\theta x \otimes \mathrm{d}x\,x
\end{aligned}$$

and

$$\begin{aligned}\Delta_L(\theta\,\mathsf{d}\theta) &= \Delta(\theta)\phi_L(\mathsf{d}\theta)\\ &= Y\phi_L(\mathsf{d}\theta)\Delta(\theta) + (B - jY + qj^2C)\theta x \otimes \mathsf{d}x\,\theta\\ &\quad + j(X - jY)\theta^2 \otimes \mathsf{d}x\,x + (A + qj^2D - qj^2Y)x\theta \otimes \mathsf{d}\theta\,x\\ &= Y\Delta_L(\mathsf{d}\theta\,\theta) + (B - jY + qj^2C)\theta x \otimes \mathsf{d}x\,\theta\\ &\quad + j(X - jY)\theta^2 \otimes \mathsf{d}x\,x + (A + qj^2D - qj^2Y)x\theta \otimes \mathsf{d}\theta\,x.\end{aligned}$$

We see from the last three relations that in order to have left covariance D must be zero. Then with Y arbitrary

$$\begin{aligned}&A = j^2qY &&C = q^{-1}Y\\ &B = j(1 - j)Y &&X = jY.\end{aligned} \tag{16}$$

On the other hand, the action of d on $\theta^3 = 0$ gives

$$1 + jY + j^2Y^2 = 0. \tag{17a}$$

So,

$$Y = j \quad \text{or} \quad Y = j^2. \tag{17b}$$

For $Y = j^2$, relations (11a) are not homogeneous. Hence, we must take $Y = j$.

Also, since

$$\begin{aligned}\Delta_L(\mathsf{d}x\,\mathsf{d}\theta) &= \Delta_L(\mathsf{d}x)\Delta_L(\mathsf{d}\theta)\\ &= F\Delta_L(\mathsf{d}\theta)\Delta_L(\mathsf{d}x) + j(qj - F)\theta x \otimes (\mathsf{d}x)^2\end{aligned}$$

we must have

$$F - qj = 0. \tag{18}$$

Here, we used that

$$(x \otimes \mathsf{d}x)(\theta \otimes \mathsf{d}x) = jx\theta \otimes (\mathsf{d}x)^2.$$

Relations (10) and (11) are explicitly as follows: the commutation relations of variables and their differentials are

$$\begin{aligned}x\,\mathsf{d}x &= j^2\,\mathsf{d}x\,x\\ x\,\mathsf{d}\theta &= q\,\mathsf{d}\theta\,x + (j^2 - 1)\,\mathsf{d}x\,\theta\\ \theta\,\mathsf{d}x &= jq^{-1}\,\mathsf{d}x\,\theta\\ \theta\,\mathsf{d}\theta &= j\,\mathsf{d}\theta\,\theta\end{aligned} \tag{19}$$

and among those first-order differentials are

$$\mathsf{d}x\,\mathsf{d}\theta = jq\,\mathsf{d}\theta\,\mathsf{d}x \qquad (\mathsf{d}x)^3 = 0. \tag{20}$$

The commutation relations between variables and second-order differentials are

$$\begin{aligned}x\,\mathsf{d}^2x &= j^2\,\mathsf{d}^2x\,x\\ x\,\mathsf{d}^2\theta &= q\,\mathsf{d}^2\theta\,x + (j^2 - 1)\,\mathsf{d}^2x\,\theta\\ \theta\,\mathsf{d}^2x &= q^{-1}\,\mathsf{d}^2x\,\theta\\ \theta\,\mathsf{d}^2\theta &= \mathsf{d}^2\theta\,\theta.\end{aligned} \tag{21}$$

The commutation relations between first-order and second-order differentials are

$$\begin{aligned}\mathsf{d}x\,\mathsf{d}^2x &= j^{-2}\,\mathsf{d}^2x\,\mathsf{d}x\\ \mathsf{d}x\,\mathsf{d}^2\theta &= q\,\mathsf{d}^2\theta\,\mathsf{d}x + (j - j^{-1})\,\mathsf{d}^2x\,\mathsf{d}\theta\\ \mathsf{d}\theta\,\mathsf{d}^2x &= j^2q^{-1}\,\mathsf{d}^2x\,\mathsf{d}\theta\\ \mathsf{d}\theta\,\mathsf{d}^2\theta &= \mathsf{d}^2\theta\,\mathsf{d}\theta\end{aligned} \tag{22}$$

and those among the second-order differentials are

$$\mathsf{d}^2x\,\mathsf{d}^2\theta = j^2q\,\mathsf{d}^2\theta\,\mathsf{d}^2x. \tag{23}$$

Now, it can be checked that the linear map Δ_L leaves invariant relations (19)–(23). One can also check that the following identities are satisfied:

$$(\mathrm{id}\otimes\Delta_L)\circ\Delta_L = (\Delta\otimes\mathrm{id})\circ\Delta_L \qquad m\circ(\epsilon\otimes\mathrm{id})\circ\Delta_L = \mathrm{id}. \tag{24}$$

We call as left coaction the map Δ_L. The map Δ_L makes the $\Omega(\mathcal{A})$ a left $\mathcal{A}$-module. So, the pair $(\Omega(\mathcal{A}), \Delta_L)$ is a left-covariant left $\mathcal{A}$-module over Hopf algebra $\mathcal{A}$. However the pair $(\Omega(\mathcal{A}),\ \mathsf{d})$ is a differential calculus over $\mathcal{A}$, and d is a left comodule map, i.e. for all $a\in\mathcal{A}$,

$$(\tau\otimes\ \mathsf{d})\circ\Delta(a) = \Delta_L(\mathsf{d}a). \tag{25}$$

Consequently, the triple $(\Omega(\mathcal{A}),\ \mathsf{d},\ \Delta_L)$ is a left-covariant differential calculus over the Hopf algebra $\mathcal{A}$.

4. Cartan–Maurer 1-forms on $\mathcal{A}$

In this section, we shall define 2-forms using the generators of $\mathcal{A}$ and show that they are left-invariant. If we call them w and u then one can define them as follows [13]:

$$w = \mathsf{d}x\,x^{-1} \qquad u = \mathsf{d}\theta\,x^{-1} - \mathsf{d}x\,x^{-1}\theta x^{-1}. \tag{26}$$

The elements w and u with the generators of $\mathcal{A}$ satisfy the following rules:

$$\begin{aligned} xw &= j^2wx \qquad & \theta w &= jw\theta \\ xu &= qux \qquad & \theta u &= jqu\theta. \end{aligned} \tag{27}$$

The first-order differentials with 1-forms satisfy the following relations:

$$\begin{aligned} &w\,\mathsf{d}x = j\,\mathsf{d}x\,w \qquad u\,\mathsf{d}x = q^{-1}\,\mathsf{d}x\,u \\ &w\,\mathsf{d}\theta = j^2\,\mathsf{d}\theta\,w + q^{-1}(1-j)\,\mathsf{d}x\,u \\ &u\,\mathsf{d}\theta = q^{-1}\,\mathsf{d}\theta\,u + q^{-2}(1-j)\,\mathsf{d}x\,u\,\theta x^{-1} \end{aligned} \tag{28a}$$

and with second-order differentials

$$\begin{aligned} &w\,\mathsf{d}^2x = j^2\,\mathsf{d}^2x\,w \qquad u\,\mathsf{d}^2x = q^{-1}\,\mathsf{d}^2x\,u \\ &w\,\mathsf{d}^2\theta = \mathsf{d}^2\theta\,w + q^{-1}(j-j^{-1})\,\mathsf{d}^2x\,u \\ &u\,\mathsf{d}^2\theta = q^{-1}\,\mathsf{d}^2\theta\,u + q^{-2}(1-j)\,\mathsf{d}^2x\,u\,\theta\,x^{-1}. \end{aligned} \tag{28b}$$

The commutation rules of the elements w and u are

$$w^3 = 0 \qquad wu = uw. \tag{29}$$

The elements w and u are both left-invariant with the following structures:

$$\Delta_L(w) = 1\otimes w \qquad \Delta_L(u) = 1\otimes u. \tag{30}$$

The counit ϵ is given by [13]

$$\epsilon(w) = 0 \qquad \epsilon(u) = 0 \tag{31}$$

and the coinverse S is defined by

$$S(w) = -w \qquad S(u) = -u. \tag{32}$$

One can easily check that the following properties are satisfied:

$$\begin{aligned} &(\mathrm{id}\otimes\Delta_L)\circ\Delta_L = (\Delta\otimes\mathrm{id})\circ\Delta_L \\ &m\circ(\epsilon\otimes\mathrm{id})\circ\Delta_L = \mathrm{id} \\ &m\circ(S\otimes\mathrm{id})\circ\Delta_L = \mathrm{id}. \end{aligned}$$

Note that the commutation relations (27)–(29) are compatible with Δ_L, ϵ and S, in the sense that $\Delta_L(xw) = \Delta(x)\Delta_L(w) = j^2\Delta_L(wx)$, $\Delta_L(w^3) = 0$ and so on.

5. Quantum Lie superalgebra

The commutation relations of Cartan–Maurer forms allow us to construct the algebra of the generators. In order to obtain the quantum Lie superalgebra of the algebra generators, we first write the Cartan–Maurer forms as

$$\mathsf{d}x = wx \qquad \mathsf{d}\theta = w\theta + ux. \tag{33}$$

The differential d can then be expressed in the form

$$\mathsf{d} = wT + u\nabla. \tag{34}$$

Here T and ∇ are the quantum Lie superalgebra generators. We now shall obtain the commutation relations of these generators. Considering an arbitrary function f of the coordinates of the Z_3-graded quantum superplane and using that $\mathsf{d}^3 = 0$ one has

$$\mathsf{d}^2 f = (\mathsf{d}w)Tf + (\mathsf{d}u)\nabla f + jw\,\mathsf{d}Tf + j^2u\,\mathsf{d}\nabla f$$

and

$$\mathsf{d}^3 f = j^{-1}w\,\mathsf{d}^2Tf + ju\,\mathsf{d}^2\nabla f - \mathsf{d}w\,\mathsf{d}Tf - j\,\mathsf{d}u\,\mathsf{d}\nabla f + \mathsf{d}^2wTf + \mathsf{d}^2u\nabla f.$$

So we need the 2-forms. Applying the exterior differential d to relations (26) one has

$$\begin{aligned} \mathsf{d}w &= \mathsf{d}^2x\,x^{-1} - jw^2 \\ \mathsf{d}u &= \mathsf{d}^2\theta\,x^{-1} - \mathsf{d}^2xx^{-1}\theta x^{-1} + uw. \end{aligned} \tag{35}$$

Also, since

$$\begin{aligned} &w\,\mathsf{d}w = j\,\mathsf{d}w\,w \\ &w\,\mathsf{d}u = j^2\,\mathsf{d}u\,w + (j - j^{-1})\,\mathsf{d}w\,u \\ &u\,\mathsf{d}w = \mathsf{d}w\,u \qquad u\,\mathsf{d}u = \mathsf{d}u\,u \end{aligned}$$

we have

$$\mathsf{d}^2w = 0 \qquad \mathsf{d}^2u = 0. \tag{36}$$

Using the Cartan–Maurer equations we find the following commutation relations for the quantum Lie superalgebra:

$$T\nabla = \nabla T \qquad \nabla^3 = 0. \tag{37}$$

The commutation relations (37) of the algebra generators should be consistent with monomials of the coordinates of the Z_3-graded quantum superplane. To do this, we evaluate the commutation relations between the generators of algebra and the coordinates. The commutation relations of the generators with the coordinates can be extracted from the Z_3-graded Leibniz rule:

$$\begin{aligned} \mathsf{d}(xf) &= (\mathsf{d}x)f + x(\mathsf{d}f) \\ &= w(x + j^2xT)f + u(qx\nabla)f \\ &= (wT + u\nabla)xf \end{aligned} \tag{38}$$

and

$$\begin{aligned} \mathsf{d}(\theta f) &= (\mathsf{d}\theta)f + j\theta(\mathsf{d}f) \\ &= w(\theta + j^2\theta T)f + u(x + qj^2\theta\nabla)f \\ &= (wT + u\nabla)\theta f. \end{aligned} \tag{39}$$

This yields

$$\begin{aligned} &Tx = x + j^2xT \qquad &&T\theta = \theta + j^2\theta T \\ &\nabla x = qx\nabla \qquad &&\nabla\theta = x + qj^2\theta\nabla. \end{aligned} \tag{40}$$

We know that the differential operator d satisfies the Z_3-graded Leibniz rule. Therefore, the generators T and ∇ are endowed with a natural coproduct. To find them, we need the following commutation relations:

$$Tx^m = \frac{1-j^{2m}}{1-j^2}x^m + j^{2m}x^mT \tag{41}$$

$$\nabla x^m = q^m x^m \nabla \tag{42}$$

where use was made of (40). Relation (41) is understood as an operator equation. This implies that when T acts on arbitrary monomials $x^m\theta$,

$$T(x^m\theta) = \frac{1-j^{2m+2}}{1-j^2}(x^m\theta) + j^{2m+2}(x^m\theta)T \tag{43}$$

from which we obtain

$$T = \frac{1-j^{2N}}{1-j^2} \tag{44}$$

where N is a number operator acting on a monomial as

$$N(x^m\theta) = (m+1)x^m\theta. \tag{45}$$

We also have

$$\nabla(x^m\theta) = q^m x^{m+1} + j^2 q^{m+1}(x^m\theta)\nabla. \tag{46}$$

So, applying the Z_3-graded Leibniz rule to the product of functions f and g, we write

$$\mathrm{d}(fg) = [(wT + u\nabla)f]g + j^{\mathrm{grad}(f)}f(wT + u\nabla)g \tag{47}$$

with the help of (34). From the commutation relations of the Cartan–Maurer forms with the coordinates of the Z_3-graded quantum superplane, we can compute the corresponding relations of w and u with functions of the coordinates. From (27) we have

$$fw = j^{2N-1}wf \qquad fu = jq^N uf \tag{48}$$

where $f = x^m\theta$. Inserting (48) in (47) and equating coefficients of the Cartan–Maurer forms, we get

$$\begin{aligned} T(fg) &= (Tf)g + j^{\mathrm{grad}(f)}j^{2N-1}f(Tg) \\ \nabla(fg) &= (\nabla f)g + j^{\mathrm{grad}(f)}jq^N f(\nabla g). \end{aligned} \tag{49}$$

Consequently, we have the coproduct

$$\begin{aligned} \Delta(T) &= T \otimes 1 + j^{-N} \otimes T \\ \Delta(\nabla) &= \nabla \otimes 1 + j^2 q^N \otimes \nabla. \end{aligned} \tag{50}$$

6. Conclusion

To conclude, we introduce here commutation relations between the coordinates of the Z_3-graded quantum superplane and their partial derivatives and thus illustrate the connection between the relations in section 5, and the relations which will now be obtained.

To proceed, let us obtain the relations of the coordinates with their partial derivatives. We know that the exterior differential d can be expressed in the form

$$\mathrm{d}f = (\mathrm{d}x\,\partial_x + \mathrm{d}\theta\,\partial_\theta)f. \tag{51}$$

Then, for example,

$$\begin{aligned}\mathsf{d}(xf) &= \mathsf{d}x f + x\,\mathsf{d}f\\ &= \mathsf{d}x\left[1 + j^2 x\partial_x + (j^2-1)\theta\partial_\theta\right] f + q\,\mathsf{d}\theta\, x\partial_\theta f\\ &= (\mathsf{d}x\,\partial_x x + \mathsf{d}\theta\,\partial_\theta x) f\end{aligned}$$

so that

$$\begin{aligned}\partial_x x &= 1 + j^2 x\partial_x + (j^2-1)\theta\partial_\theta &\qquad \partial_x\theta &= j^2 q^{-1}\theta\partial_x\\ \partial_\theta x &= qx\partial_\theta &\qquad \partial_\theta\theta &= 1 + j^2\theta\partial_\theta.\end{aligned} \tag{52}$$

The commutation relations between derivatives are

$$\partial_x\partial_\theta = jq\partial_\theta\partial_x \qquad \partial_\theta^3 = 0. \tag{53}$$

The Z_3-graded Hopf algebra structure for ∂ is given by

$$\begin{aligned}\Delta(\partial_x) &= \partial_x\otimes\partial_x &\qquad \Delta(\partial_\theta) &= \partial_\theta\otimes\partial_x + \partial_x\otimes\partial_\theta\\ \epsilon(\partial_x) &= 1 &\qquad \epsilon(\partial_\theta) &= 0\\ S(\partial_x) &= \partial_x^{-1} &\qquad S(\partial_\theta) &= -\partial_x^{-1}\partial_\theta\partial_x^{-1}\end{aligned} \tag{54}$$

provided that the formal inverse ∂_x^{-1} exists. However, these comaps do not leave invariant relations (52).

We know, from section 5, that the exterior differential d can be expressed in form (34), which we repeat here,

$$\mathsf{d}f = (wT + u\nabla)f. \tag{55}$$

Considering (51) together with (55) and using (33) one has

$$T = x\partial_x + \theta\partial_\theta \qquad \nabla = x\partial_\theta. \tag{56}$$

Using relations (52) and (53) one can check that the relations of the generators in (56) coincide with (37). It can also be verified that the action of the generators in (56) on the coordinates coincides with (40).

The Z_3-graded noncommutative differential geometry we have constructed satisfies all expectations for such a structure. In particular, all Hopf algebra axioms are satisfied without any modification. Moreover, the extension of the structure presented in this paper can be generalized to Z_N.

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Appendix. Quantum matrices in the Z_3-graded space

In this appendix, we shall investigate the quantum supermatrices in the Z_3-graded quantum superplane. We know, from section 2, that the Z_3-graded quantum superplane is generated by coordinates x and θ, and commutation rules (1), which we repeat here,

$$x\theta = q\theta x \qquad \theta^3 = 0. \tag{57}$$

These relations define a deformation of the algebra of functions on the superplane generated by x and θ, and we have denoted it by $\mathcal{A}$. The dual Z_3-graded quantum superplane $\mathcal{A}^\star$ is generated by φ and y with the relations

$$\varphi y = qjy\varphi \qquad \varphi^3 = 0 \tag{58}$$

where $\mathsf{d}x = \varphi$ and $\mathsf{d}\theta = y$ in (20).

Let T be a 2×2 (super)matrix in Z$_3$-graded space,

$$T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \tag{59}$$

where a and d with respect to the Z$_3$-grading are of grade 0, and β and γ with respect to the Z$_3$-grading are of grade 2 and of grade 1, respectively. We now consider linear transformations with the following properties [16]:

$$T : \mathcal{A} \to \mathcal{A} \qquad T : \mathcal{A}^\star \to \mathcal{A}^\star. \tag{60}$$

The action on the elements of $\mathcal{A}$ of T is $\left(\begin{smallmatrix} x' \\ \theta' \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & \beta \\ \gamma & d \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ \theta \end{smallmatrix}\right)$. We assume that the entries of T are j-commutative with the elements of $\mathcal{A}$, i.e. for example,

$$ax = xa \qquad \theta\beta = j^2\beta\theta$$

etc. As a consequence of the linear transformations in (60) the elements

$$x' = ax + \beta\theta \qquad \theta' = \gamma x + d\theta \tag{61}$$

should satisfy relations (57). Applying the exterior differential d to relation (61) one has

$$\varphi' = a\varphi + j^2\beta y \qquad y' = j\gamma\varphi + dy. \tag{62}$$

These elements must satisfy relations (58). Consequently, we have the following commutation relations between the matrix elements of T:

$$\begin{array}{lll} a\beta = j^{-1}q^{-1}\beta a & d\beta = jq^{-1}\beta d & \\ a\gamma = q\gamma a & d\gamma = q\gamma d & \\ ad = da + q^{-1}(1-j)\beta\gamma & \beta\gamma = q^2\gamma\beta & \gamma^3 = 0. \end{array} \tag{63}$$

We shall denote with $\mathrm{GL}_{q,j}(1|1)$ the quantum supergroup in Z$_3$-graded space determined by generators a, β, γ, d satisfying the commutation relations (63).

Note that these relations can be obtained from the requirement that $\mathcal{A}$ and $\mathcal{A}^\star$ have to be covariant under the left coactions

$$\delta : \mathcal{A} \to \mathrm{GL}_{q,j}(1|1) \otimes \mathcal{A} \qquad \delta^\star : \mathcal{A}^\star \to \mathrm{GL}_{q,j}(1|1) \otimes \mathcal{A}^\star \tag{64}$$

such that

$$\begin{array}{ll} \delta(x) = a \otimes x + \beta \otimes \theta & \delta(\varphi) = a \otimes \varphi + j^2\beta \otimes y \\ \delta(\theta) = \gamma \otimes x + d \otimes \theta & \delta(y) = j\gamma \otimes \varphi + d \otimes y \end{array} \tag{65}$$

provided that the entries of T are j-commutative with the elements of $\mathcal{A}$ and $\mathcal{A}^\star$.

Note that relations (63) are slightly different from the results of [12]. The reason for this difference is that in [12], since it was assumed that commutation relations of the differentials are

$$\mathsf{d}x\,\mathsf{d}\theta = r^{-1}\,\mathsf{d}\theta\,\mathsf{d}x \qquad (\mathsf{d}x)^2 = 0 = (\mathsf{d}\theta)^2$$

the commutation relations among the matrix elements of a matrix in Z$_3$-graded space were obtained via the use of them. On the other hand, we use the commutation relations of the coordinates of the Z$_3$-graded quantum superplane with their differentials.

An interesting problem is the construction of a differential calculus on the Z$_3$-graded quantum supergroup $\mathrm{GL}_{q,j}(1|1)$ using the methods of this paper and [17]. Work on this issue is in progress.

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